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ABSTRACT
Examined are some of the interrelations among three areas--first, linguistic work inspired by the ideas of Zellig Harris--second, logical investigations concerning the nature of mathematical reasoning--and third, mathematical education. The author states that his main concern is to bring some basic linguistic concepts and hypotheses to the study of deductive reasoning and, then, to suggest applications to mathematical education. The substance of this paper is divided in three parts. Part I contains some introductory remarks delimiting the author's understanding of what mathematical reasoning is. In addition, this section relates the author's structure of mathematical discourse to the structure of English discourse. In Part II, the author outlines the nature of a theory of proof and suggests the utility of such a theory for mathematical education. In Part III, the author develops several basic ideas involved in devloping a usable theory of proof. (RP)

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PRELIMINARY DRAFT

DISCOURSE GRAMMARS AND THE STRUCTURE OF MATHEMATICAL REASONING

by

John Corcoran

In this paper I will examine some of the interrelations among three areas: first, linguistic work inspired by the ideas of Zellig Harris; second, logical investigations concerning the nature of mathematical reasoning; and third, mathematical education. My main concern is to bring some basic linguistic concepts and hypotheses to the study of deductive reasoning and, then, to suggest applications to mathematical education. I take mathematical education to be the area of study which attempts to better understand teaching and learning of mathematics and also to improve mathematical teaching in practice.

The substance of this paper is divided in three. Part I contains some introductory remarks delimiting in broad strokes my understanding of what mathematical reasoning is. In addition I relate the structure of mathematical discourse to the structure of English discourse. In Part II, I try to outline the nature of a theory of proof and also to suggest the utility of such a theory for mathematical education. In Part III I develop several basic ideas involved in developing a usable theory of proof.

For readers with a practical interest in mathematical education, the central core of the paper is the last section of Part II where I discuss the utility of a theory of proof for mathematical education. From this point of view, the discussion preceding the central core provides the background needed to understand what a theory of proof is, while the discussion following the central core develops several basic ideas essential to the construction of such a theory.

PART I

MATHEMATICAL REASONING AND THE STRUCTURE OF LANGUAGE

--Logical Consequence and Inference

Mathematical reasoning, or deductive reasoning, is a process whereby a person comes to understand, to know, that if certain statements were true then a certain other statement would necessarily also be true. When a man thinks through the axioms and a certain theorem in geometry and comes to know that if the axioms are true then the theorem must also be true, then that man is engaging in mathematical reasoning. In giving a proof a man expresses his reasoning in writing or in speech. Thus reasoning is a mental process whereas a proof is a linguistic entity, written or spoken. Apparently we have no access to another person's mathematical reasoning except through the proofs he offers. Thus proofs provide "the data" in the study of mathematical reasoning.

In the above paragraph I have implicitly presupposed familiarity with three special concepts: Logical consequence, deductive reasoning and proofs. A brief review of these ideas may prove helpful because each of them plays a prominent role in subsequent developments.

Consider a given set of statements (axioms) and a given single statement. It may happen that the given statement follows logically from the axioms. That is, it may happen that if the axioms were all true then the given statement would necessarily also be true. In this case the

given statement is a logical consequence of the axioms; the axioms imply the given statement. Naturally, even if the given statement does follow there is no guarantee that anyone knows that it follows (otherwise there would be no open questions in mathematics).

In order to know that a given statement follows from given axioms, it is necessary to reason deductively from the axioms to the given statement, to trace out the logical steps whereby one comes to know that if the given axioms are true, the given statement must also be true. Thus, deductive reasoning is human activity; it is a mental process subsumed under the broader heading of learning. In a particular case of deductive reasoning, a person learns something, comes to know something, viz., that a certain statement is a logical consequence of certain other statements.

A proof is an articulation or expression of deductive reasoning. In giving a proof of a given statement from given axioms one expresses his reasoning why the given statement follows. Thus a proof is something which can be written or spoken--a proof is a linguistic element similar in many ways to narrative paragraphs.

There is a metaphor which neatly separates the three ideas. A logical consequence connection is a path in logical space; when we reason we trace the path; and a proof is a set of directions we give to others for retracing the path. The metaphor breaks down when indirect reasoning (reductio ad absurdum) is considered because "tracing a path" from

axioms and a denial of a statement to a contradiction does not seem to be "tracing a path" from the axioms to the statement. There are doubtless other more serious deficiencies in the metaphor.

It is important to distinguish deductive reasoning from the perhaps more creative process of discovering a logical connection. Imagine, for example, that you are given a set of axioms. You read them carefully and understand them and then you "see" that something else follows. This "seeing" is actually a kind of guessing because you do not know that the additional statement follows until you verify it by step-by-step reasoning. The word "infer" is frequently used to cover both the initial guessing and the subsequent deductive reasoning. According to the way that I am using the words "deductive reasoning" one usually does not reason until he has already guessed a possible consequence. In addition, I should emphasize that deductive reasoning is also involved in "following" a proof, i.e., in seeing for oneself that it shows why the conclusion follows. Naturally, the reasoning involved in following a proof is not nearly as creative as the reasoning involved in discovering the exact logical connection--but it is, nevertheless, deductive reasoning.

The reader deserves to be warned that the word "proof" is ambiguous in normal mathematical parlance. It is certainly used in the above sense to indicate an articulation of dedutive reasoning, but it is also used in the sense of a

particular logical connection or path in logical space or something of the sort. For example, we speak of trying to discover a new proof of a known theorem (from given axioms). Here we are not looking for a new way of describing the known path of reasoning, but we actually want a new path of reasoning--a new way of getting to the theorem from the axioms. The reason that we would want a new path would be that we find the known one to be devious, round-about, overly intricate, unnatural--in a word, inelegant.

It is interesting to notice that the distinction between the relation of logical consequence and the act of inferring is already implicit in non-technical English. It is grammatically acceptable to say that one statement implies another statement, but it is not acceptable to say that one statement infers another statement. On the other hand, one can say that a person infers one statement from another statement.

At this point we have distinguished logical consequence, an objective logical relation, from deductive reasoning, a human activity. Deductive reasoning is one of the primary activities of mathematicians because mathematicians are concerned to establish logical connections between axioms and other statements. Since reasoning is a human activity it should be expected that some people are more skilled in it than others. It is almost by definition that a good mathematician is more skilled in deductive reasoning than a poor one. I say "almost" because I have heard of

mathematicians who are unskilled theorem-provers, but who have gained reputations for being able to guess new theorems with uncanny accuracy. In any case, after a statement has been guessed to follow, one may go through the process of reasoning step-by-step why (or how) it follows. The verbal or written articulation of the reasoning is a proof. Given a written proof, one can retrace the steps of reasoning expressed in it and rediscover (or see for himself) that the conclusion follows.

Next I will change the subject from logic to linguistics in order to discuss levels of language. Afterward I will bring the two together.

--Levels of Language

Stratification: The notion that language is stratified into increasingly complex levels of organization is clearly reflected in our writing system. A written text is organized in paragraphs. The paragraphs are naturally segmented into sentences. Sentences are perceived as composed of phrases which in turn are composed of words, and the words are strings of letters. The lowest level of the written language is the alphabet. Next, we have the level of words, then the level of phrases, then sentences and, perhaps finally, the level of discourse which contains paragraphs and "texts." For illustrative purposes, let us assume that English is stratified according to the above scheme. It seems to me obvious that English is stratified in some way or other, but it is at

least doubtful whether it is stratified even roughly in accord with the above scheme.

By "language" the linguist means the spoken language. Thus, we are assuming that the above stratification applies to the spoken language.

Let us introduce terminology appropriate for a detailed discussion of the assumed stratification of English. The alphabet of (spoken) English is the set of basic, "indivisible" spoken symbols. The objects in the spoken alphabet are generally called phonemes and, in a written alphabet representing phonemes, the representations of, e.g., "early" and "yearly" would not suggest that they rhymed, whereas the representations of "sax" and "sacks" would be the same. The lexicon of English is the set of words of English. For the following we do not introduce any special terminology: the set of phrases, the set of sentences, the set of discourses. The corresponding levels are called respectively: alphabetic, lexicographic, phraseological, sentential and discourse.

Reality of Language Structure: One very important theoretical question in linguistics concerns the "reality" of the stratification into levels. For example, could it not be the case that the stratification is only a structure which we find convenient to impose on English but which corresponds to nothing real in English? Many linguists do not regard this as a substantive question because, some reason, "Either there is a 'real' structure to English or there is

not. If there is a 'real' structure, then, presumably, if we work hard enough and are flexible and imaginative, then the structure that we find most convenient will correspond exactly to the real structure. Thus, convenience is the important criterion. On the other hand, if there is no 'real' structure, then what else is there besides convenience?"

There is additional debate concerning which levels are "real" and what kinds of reasons are relevant to determining the "reality" of a level. Some linguistic work suggests that some relevant evidence can be gleaned from studies of the patterns of stress and intonation and of the co-occurrence restrictions in actual speech. Other linguists (Chomsky, Section 8.1) have made interesting suggestions concerning ways of justifying an intermediate level, B, given the existence of two levels A and C.

In my opinion, all of the above questions are very important and very difficult. But they are really out of place in this paper except that they may have started to bother the perceptive and critical reader. My brief remarks are intended as an acknowledgement of the difficulties.

Internal Structure of Levels: Now, we wish to consider the internal structure of each of the five levels. What we mean by the internal structure of a level is merely how the elements of that level are interrelated. We disregard any possible internal structure on the alphabetic level because we are considering the phonemes (or letters) to be indivisible units.

Generalizing on Harris' ideas (1951, pp. vi, vii), we postulate a "kernel/compound" structure at each of the higher levels. This means that each element in a given level is either a simple element or a combination of simple elements. The set of simple elements of a level is called the kernel of that level, the non-kernel elements are called compounds. The kernel on the lexicographic level might contain 'black' 'bird' and 'like', while "birds-like", "unbird-like", and 'blackbird' might be among the compounds. The kernel on the sentential level might include 'Birds fly' and 'Fish swim', while 'Birds swim and fish fly' and 'Birds do not fly' would be compounds. On the discourse level the kernel would contain only sentences which can stand alone. For example, 'Birds fly' would be in the ker[illegible] whereas 'They fly' would not. Among the compound discourses would be things like the following: 'Birds don't swim. They fly.'

Notice two things. First, in each case, kernel elements were constructed directly only using elements on the lower levels. Second, compounds were composed of kernel elements, using only lower level elements as "connections". To say that this is true, in general, is most likely wrong. However, for the sake of our illustration, we are making this assumption.

The total set of assumptions that we have made about English amounts to something approaching the simplest n[illegible] trivial hypothesis about the structure of English--and as

such it is almost certainly wrong. Indeed, many linguists will be offended by the thought that these assumptions could be seriously entertained as possible. Thus, I want to emphasize that the assumptions are made in a pedagogical spirit only.

Let us briefly review the assumptions. First, we assumed that English is stratified into five levels roughly corresponding to letters, words, phrases, sentences and discourses. Second, we assumed that each level above the alphabetic has a kernel/compound structure. Third, we assumed that each of the higher levels is somehow obtained by combining only elements on the same or lower levels. Our third assumption is meant to imply that, given a description of the spoken alphabet of phonemes the following hold: first, that the words can be described without reference to phrases, sentences, or discourses; second, that the phrases can be described without reference to sentences or discourses; and third, that the sentences can be described without reference to discourses. It is the last assumption that will be found most obnoxious and, I must admit, I do not think that it is very plausible myself.

It is interesting to compare the levels of language with meanings as "perceived" in written language. On the lowest level, we have units which are perceived as written language (as opposed to mere marks), but which are not necessarily meaningful as such. For example, the letter 'p' is not meaningful, but the letter 'a' can be meaningful.

Next, we have words which are definitely meaningful. The kernel words can be thought of as words which do not have other words as parts. Actually one kernel word could have other words as parts, literally, but it would still be regarded as a kernel word if its meaning was not related in any way to the meaning of its part. For example, 'dog' has 'do' for a part, but the meaning of 'dog' is in no way related to the meaning of 'do'. Naturally some words are composed of only one phoneme, e.g., 'a' and 'I'. Thus, the lexicographic level (or word level) is the primary level as far as meaning is concerned. As far as meaning is concerned, the alphabetic level is dispensible--we utter words and it is accidental that they are made up of phonemes (letters). Theoretically we could have a language in which each word was a single phoneme so that the lexicographic and alphabetic levels would be the same. The trouble with this is that aft a certain number of words were introduced into the language we would have to have extraordinarily sharp ears (and "sharp" tongues) to communicate.

One interesting thing about the phraseological level is that the meanings of complex phrases are very dependent on the meanings of the words which are their parts, e.g., the meaning of 'the king of Iowa' is certainly dependent on the meanings of the words in it. At this level, however, the meanings are still in a sense word meanings--they are not yet sentential in nature. They are not true or false, for example. We correctly speak of noun phrases, verb phrases,

adjective phrases, and adverbial phrases--indicating that the meanings of phrases are "functionally" the same as the meanings of words.

On the sentential level a new kind of meaning emerges. Indeed, it might be said that communication begins on the sentential level because although the utterance of a word or phrase may permit us to know what the speaker is talking about, it will not tell us what he is saying about it. It is important to notice that the meaning we get from a sentence depends not only on the meanings we attach to the particular words in the sentence, but also on the way that we hear (see) the sentence composed of phrases. Consider the following:

You know how sincere freshmen are.

(Do you see 'sincere' grouped with 'how' or with 'freshmen'?)

On the level of discourse still another kind of meaning emerges. The point here is that the kind of thing communicated by a discourse is generally richer and more complex than the kind of thing communicated by a single sentence. In particular, a declarative sentence could be said to communicate a "fact", whereas a paragraph composed of declarative sentences could communicate facts together with an organization of them. It is even more instructive to consider proofs as examples of discourses (Harris, 1954). What is communicated in a proof is the reasoning from its assumptions to its conclusion and generally none of the sentences in a proof are asserted as declarative sentences. We can reason

from admittedly false assumptions to an admittedly false conclusion and communicate the reasoning in a correct proof--necessarily not asserting (as facts) any of the sentences in the proof.

Some linguists have suggested that the meaning of a discourse is merely the meaning of the logical conjunction of all the sentences in the discourse. This is obviously not the case because if it were so, then the order of occurrence of sentences in a discourse would be irrelevant. It is clear that the order of occurrence of sentences in a discourse is of crucial importance with regard to the meaning of a discourse. As an experiment one might permute the sentences of a given paragraph and then see if the result means the same as the original. (For example, try interchanging the first and third sentences of this paragraph.)

Most of the linguistics before the 50's was focused primarily below the sentential level and much of the work at the sentential level was piecemeal and (therefore?) unexciting to persons with a mathematical outlook. The interesting developments started in the 50's with Harris' investigations above the sentential level. As a result of Harris' work, the sentential level was investigated in a more systematic and mathematically interesting fashion. Harris had become interested in the obvious fact that certain stretches of speech composed of several sentences have a definite kind of structure not reducible to sentential structure. These structured stretches of speech (or writing) were called discourses.

So far we have distinguished five levels of language: the alphabetic, the lexicographic (words), the phraseological, the sentential and the discourse levels. Each of the higher levels encompasses an increasingly complex structure which depends on the levels below it. When we apply the conceptual import of these distinctions to formalized mathematical communication, there are no new difficulties. The basic alphabet of primitive symbols provides the alphabetic level. For example, in a language designed for arithmetic we might have an alphabet containing: a symbol x and a "prime" ' for constructing variables (x, x', x'', x''', etc.), the arithmetic symbols, 0, 1, +, $\cdot$, < and the logical symbols &, v, $\sim$, $\supset$, =, etc. The lexicographic level would include 0, 1 and the compound symbols obtained by "priming" the symbol x (i.e., the variables). On the phraseological level we would have all of the elements just mentioned together with the other terms which enter equations, viz., (0+1), ((0+1) $\cdot$1), (x' + x'''), ((0+x'') $\cdot$x'''), etc. The sentential level would contain: the equations, ((0+1) = x), etc.; the inequalities (((0+1)+1)<1), etc. and all of the compound formulas made up from equations and inequalities by use of quantifiers and connectives. Finally, the discourse level would include the proofs.

Thus we have an analogy between the levels of English and the levels of a language of arithmetic according to which the discourses of English correspond to the proofs in the arithmetic language. One striking difference is that in

English there are many different kinds of discourses (narrative, descriptive, explanatory, demonstrative, etc.), whereas in the arithmetic language apparently there is only one kind of discourse--the proof. Of course, the same sort of thing is true on the sentential level (all formulas in arithmetic are declarative) as well as on the phraseological level (all phrases in arithmetic are noun phrases such as (0 + 1)).

--Grammars

One of the main projects of modern linguists is to give a complete and systematic description of the entire English language. Linguists and philosophers have suggested that English be regarded as two separate but interrelated systems, a syntactical system (or system of symbols regarded abstractly) and a semantical system (or system of meanings). A description of the syntactical system of a language is called a grammar of that language.

It is reasonable to require that any adequate grammar of English must consist in adequate descriptions of each level in terms of the preceding level (or levels) together with whatever other concepts are needed. In other words, a description of the syntactical system of English should be stratified in accordance with the way that English itself is stratified. Moreover, a description of a given higher level must account for how each element of that level consists of elements of the lower level (s). Thus, a grammar

of English will consist in (1) a description of the alphabetic level, (2) a system of rules describing how the words are built up of phonemes and words, (3) a system of rules describing how the phrases are built up from phonemes, words and phrases, (4) a sententíal grammar or system of rules describing how sentences are built up, (5) a discourse grammar describing how discourses are built up. In short, a grammar of English would consist in a description of the basic alphabet together with four systems of rules each subsequent one of which depends on the lower level systems.

Without any loss of generality we can think of each of the four rule systems as including two types of rules: first, initial-string or "kernel" rules which describe the kernel; second, production rules which specify the compounds by indicating how the compounds may be constructed from the kernel elements. Rule systems of this sort are sometimes said to be in "kernel/transformation" form. Let A represent the alphabet, A_1 the set of words, A_2 the set of phrases, A_3 the set of sentences and A_4 the set of discourses. Let P_1, P_2, P_3 and P_4 be the rule sets which describe (produce) A_1, A_2, A_3 and A_4 respectively. If we indicate the levels that can be referred to in a rule set by writing the names of those levels after the names of the rules, then, according to our third assumption, a grammar of English can be represented as follows:

A		alphabet
$P_1(A)$	$\rightarrow A_1$	lexicon
$P_2(A+A_1)$	$\rightarrow A_2$	phrases
$P_3(A+A_1+A_2)$	$\rightarrow A_3$	sentences
$P_4(A+A_1+A_2+A_3)$	$\rightarrow A_4$	discourses

The arrow means "produces" or "describes". The third line can be read: a set of rules depending on the alphabet and the lexicon describes the set of phrases.

Below we give an example of a grammar of this sort to describe part of the arithmetic language mentioned above.

Partial Crammar of Arithmetic Language

A : x, ', 0, 1, +, ·, <, +, ∼, v, , &,),(, ┌, └, (alphabet)

P_1: (a) Initial Strings: 0 and 1 are constant words

(b) Initial Strings: x is a variable word

Production: if S is a variable word then S' is a variable word

A_1 : 0, 1, x, x', x'', ... (words)

P_2 : Initial Strings: all words are phrases

Productions: if S_1 and S_2 are phrases then (S_1+S_2) is a phrase.

if S_1 and S_2 are phrases then $(S_1 \cdot S_2)$ is a phrase.

A_2 : 0, 1, x, x', ..., (0+0), (0+1), (0+x), ..., ((0+1)+0), ((0+1)+1),... (phrases)

P_3 : Initial Strings: all strings $(S_1=S_2)$ and $(S_1<S_2)$ are sentences where S_1 and S_2 are phrases.

Productions: if S is a sentence then $\sim$S is a sentence

if S_1 and S_2 are sentences then $(S_1 \& S_2)$ is a sentence

if S_1 and S_2 are sentences then $(S_1 v S_2)$ is a sentence

if S_1 and S_2 are sentences then $(S_1 \supset S_2)$ is a sentence

A_3: (0=0), (0=1),..., (0<0), (0<1),..., $\sim$(0=0), $\sim$(0=1),..., $\sim$(0<0), $\sim$(0<1),... ((0=0)<(0=1))..., ((0=0)v(0=1)),... ((0<0)v(0=1)),... (sentences)

P_4 : ???

(proofs)

A_4 : ???

The above is a partial grammar of the part of the arithmetic language not involving the quantifiers. It is obvious that all such sentences are described or produced by P_3, i.e., are in A_3. P_4, which is presently left out, would generate or describe the correct proofs in this part of the arithmetic language. Before we give our opinion of what this would be like, we want to discuss the nature and value of a correct theory of proof. In the final part of the paper we will contrast twopossibilities for P_4.

Note: The above is not the only language suitable for development of arithmetic. Indeed, there are several others among which are found some which do not satisfy our three assumptions. In particular, some of them contain phrases which are constructed from sentences. For example, consider the noun phrase

"the least number greater than 2"

This would be expressed in some languages by '$(\imath x)((1+1)<x)$' which contains the sentential expression $((1+1)<x)$. Such languages exhibit an interdependence between the sentential level and the phraseological level, thus violating the third assumption which implies that phrases can be composed only of phrases, words and letters (i.e., of same or lower level elements).

There is good reason to believe that the same sort of thing is true of English and, moreover, it appears that some of the sentences in English are derived from discourses further compounding the degree to which the third assumption fails. If the latter is true, then no adequate

sentential grammar of English can be constructed until at least a part of the discourse grammar is constructed.

PART II

The Nature of A Correct Theory of Proof and Its Value

--Proofs and Rules of Inference

As we have been using the word above, a proof is an articulation of deductive reasoning from premises to conclusion. Thus when a mathematician writes a proof he is primarily interested in communicating his reasoning to others. He is explaining to others his reasoning that if the premises are true, the conclusion must also be true. Secondarily, he is recording a mental process/event -- viz. the particular process of reasoning from those particular premises to that particular conclusion during a particular time interval.

Regularity in Proofs: If we consider proofs that we have written or if we survey the proofs found in the literature of mathematics we find many repetitions of simple patterns. This is a clue to the fact that the writing of proofs is a rule-governed activity. However, if we recall our experiences we will notice that in writing proofs we do not think of ourselves as following rules. It is only after the fact that we see the patterns and postulate the existence of the rules to account for the regularity. This situation is analogous to the situation involving

writing of sentences. After seeing many examples of sentences, we notice repeating patterns and postulate the existence of rules to account for the regularity. Sentences are constructed according to rules but we are not conscious of following rules in writing sentences. The same with proofs.

When you write a proof you are generally doing (or redoing) the reasoning that you are expressing in the proof. Moreover, when you are reasoning in a particular branch of mathematics (e.g., geometry or arithmetic) you are generally thinking about the subject matter of that branch -- although, as Hilbert and others point out, if your reasoning is correct, the subject matter is irrelevant and the reasoning would apply equally well to any other subject matter. The point that I am making is that when you are writing a proof you are too busy to think of any rules even if you knew which ones to think of. This is exactly analogous to speech: when you utter a sentence you are generally thinking about what the sentence is about and thus are too busy to bother with rules. Indeed, for example, as you begin to learn a foreign language in a classroom situation, as long as you have to think of the rules you generally make rather dull conversation because you are too busy to give much thought to what you are talking about. Thus, carrying this over to reasoning, if you knew the rules explicitly and actually thought of them while you reasoned you would likely not get

very far in your mathematics.

Rules of Inference: Let us use the term "rule of inference" to refer to the rules according to which proofs are constructed. The rules of inference are rules for constructing proofs in the same way that the rules in a sentential grammar are rules for constructing sentences. Because of our hypothesis that the discourse level, which includes the proofs, must have kernel/compound structure there will be two types of rules: initial string rules asserting that certain strings are proofs ab initio and production rules which build up compound proofs from simpler ones. As a result of my own experience in formulation rules of inference it seems that each production rule can be written in the following form: if such-and-such is a proof then the result of adding so-and-so to the end of it is also a proof. This implies that each production-type rule of inference has the effect of lengthening an already existent proof.

Since proofs frequently begin with assumptions layed down without proof, we may suppose that one initial string rule simply says that any sentence may be written down to start a proof provided that it is clearly marked as an assumption. Thus we might state the initial-string rule of assumption: 'Assume p' may be written to start a proof where p is any sentence. In addition, since assumptions are also written at non-initial places in proofs we also have a production rule of assumptions: any proof may be lengthened by the addition of 'assume p'. Other production rules are

easy to think of. The rule of modus ponens can be stated: Any proof containing a sentence p and also containing a sentence 'if p then q' may be lengthened by adding q onto the end. Many other rules will come to mind.

Knowledge of Rules of Inference: It is important to distinguish a stronger and a weaker sense in which one may know a rule of inference. Let us say that a person has weak knowledge of a rule of inference if he reasons in accord with that rule. Thus weak knowledge of a rule of inference is a non-self-conscious kind of knowledge. All mathematicians and most people, I imagine, have weak knowledge of quite a few rules of inference although few people are self-conscious about the rules according to which they reason. On the other hand, let us say that a person has strong knowledge of a rule of inference if he can explain the details of the rule, point-out places where it is used, etc. Strong knowledge of a rule of inference is a very self-conscious kind of knowledge. Mathematicians generally have weak knowledge of many rules of inference and strong knowledge of very few. A logician who is poor at reasoning may have strong knowledge of many rules of inference and weak knowledge of very few, although most logicians, it seems, have weak knowledge and strong knowledge of many rules of inference.

The same distinction carries over to knowledge of rules of sentence construction. All speakers of English have weak knowledge of many sentential rules whereas only linguists can be expected to have strong knowledge of more than a few

such rules. Linguists make it their business to have strong knowledge of rules of sentence construction whereas other speakers are content to be able to use the rules, i.e., to have weak knowledge of the rules.

Naturally, it is not to be expected that everyone has even weak knowledge of all rules of inference. Certainly the high school freshman could not be expected to know all of the rules of inference used by the professional mathematician. In a sense, knowing a rule of inference amounts to understanding a type of logical connection. Of course, as people acquainted with mathematical education, we have all had the discouraging experience of seeing a student mimic a teacher's pattern of reasoning without understanding it. In such cases, I believe, we will always be able to ascertain that the student has not learned <u>the rule</u>, but only the superficial aspects of a few applications of it. Nevertheless, I must acknowledge the <u>theoretical</u> possibility of a student who knows how to use an impressibly large class of rules without understanding any of them. Such a student could write down a correct proof of a conclusion from some assumptions without believing that the conclusion actually followed from them--i.e., he would not be willing to risk anything to defend the thesis that if the assumptions were true then the conclusion would necessarily also be true.

Even though a given person may not know all of the rules of inference and, indeed, as the skills of mathematical reasoning evolve, new rules may come into use; it is most

likely the case that most normal high school freshmen know several of the simpler rules. Moreover, it is my view that some more complex rules are learned by developing skill in the use of the simpler rules and, then, seeing how steps may be skipped.

This means that after a student has gone through a certain fixed pattern of detailed reasoning several times he may develop a feel for the upshot of the pattern and begin to omit the details in future proofs--thus, in effect, gaining weak knowledge of a more complex rule. We may imagine that the professional mathematician, after years of experience in deductive reasoning, has developed weak knowledge of very complex rules well beyond the comprehension of beginning students. From this point of view, it is natural to expect that as mathematical reasoning becomes increasingly sophisticated, more and more complex rules of inference will evolve.

If we wish we may even speculate that the mathematics student has two kinds of "vocabularies" of rules--an active vocabulary that he can actually use in doing proofs and a passive vocabulary of rules which he can "follow" but not use. This sort of hypothesis may partially account for inability of students to recreate reasoning that they have followed in class.

Correctness of Rules of Inference: We may wonder about correctness and incorrectness of rules of inference--is it conceivable that a few persons or a whole society of

persons write proofs according to incorrect rules? Indeed, suppose that everyone wrote proofs according to a certain rule, would not the universal acceptance of a rule make it correct? On a certain level, these are very easy questions once we recall that a proof is designed to show that a certain conclusion follows from certain premises. If a conclusion follows from some premises then it is impossible that the premises are true and the conclusion false. Thus _if_ a system of rules could be used to prove a false sentence from a set of true sentences _then_ certainly at least one of the rules is incorrect or, as we say in logic, _unsound_. Thus, it is _possible_ that a few persons or a whole society of persons write proofs according to incorrect rules. (It is possible but I have never seen it happen--although I have seen people make mistakes in proofs.) Moreover, concerning this second question we can say that the universal acceptance of a rule of inference would not make it sound.

Incidentally, it follows from what has been said above that if a certain society writes proofs incorrectly then possibly someone could discover that fact--however, if a society writes proofs correctly then there seems to be no way of finding out for sure that it does.

Parenthetically, I might add here that if I were an Intuitionist, I would have said that I had seen examples of the use of unsound rules. The Intuitionist would say that most mathematicians use unsound rules and that much of the literature of mathematics contains incorrect proofs.

In particular, Intuitionists regard one of the forms of indirect proof as unsound. Let us consider this in a little more detail. The kind of indirect (or reductio ad absurdam) reasoning involved in the standard proof of the irrationality of $\sqrt{2}$ from the axioms of arithmetic proceeds, after the (tacit) assumption of the axioms, by assuming that $\sqrt{2} = n/m$ for some integers n and m and deducing a contradiction. This sort of reasoning is regarded as sound by the Intuitionists because what the Intuitionist means by "not p" is that the assumption of p leads to a contradiction. However the Intuitionist does not regard as sound the other reductio rule which allows one to prove p from some assumptions by assuming "not - p" and deriving a contradiction. For him this would only prove "not-not-p", from original assumptions. "Not-not-p" means that it is absurd to assume that p is absurd and, for the Intuitionist, this does not in turn mean that p itself is true. This view leads to the rejection of one rule of double negation (any proof containing "not-not-p" may be lengthened by adding p) and to the rejection of the rule of excluded middle (any proof may be lengthened by adding "p or not-p").

--Theories of Proof

By a theory of proof for English, say, I mean a discourse grammar (1) which is intended to describe some or all of the proofs expressible in English and (2) whose rules are intended to be rules of inference known by persons who

express their reasoning in English. If we are given such a theory, we may want to inquire concerning its correctness and its comprehensiveness. It would be natural to call it <u>correct</u> if each of its rules were used by some speakers of English. (There are, of course, other possibilities but this one will suffice in this context.) Furthermore, it would be natural to call it <u>comprehensive</u> if every rule used by any speaker of English were included among its rules. Of course, the correctness and the comprehensiveness of a given theory of proof would be relative to a given time in order to leave open both the possibility of "old" rules being abandoned and also the possibility of "new" rules being "devised."

The hope of ever getting a correct and comprehensive theory of proof is dim. But it is certainly possible to contribute toward such a theory. This would be done first by considering one's own reasoning and trying to formulate the rules that one actually uses himself. The next step would be to survey the mathematical literature in an attempt to find proofs that are not constructible by means of one's own rules and which, therefore, may be presumed to be constructed according to "new" rules. After some of these were formulated the continuation of the project would involve getting other workers to formulate their own rules and to help in the survey of the literature. It is hard to imagine how one could ever determine whether a particular theory were comprehensive and, of course, if a theory were

comprehensive relative to a fixed time it may very well not be comprehensive relative to a later time.

To many readers the above will sound at least utopian if not far-fetched. It may very well be utopian but, given the Chomsky-Harris idea of trying to develop a sentential grammar of English, the above can easily be seen as an application of the same core idea to a <u>part</u> of the totality of English discourses. Thus, the idea of a comprehensive discourse grammar for all of English is even more utopian. Now, as for being far-fetched, I would simply reply that it is no more f[illegible]etched than the ideal of a comprehensive sentential gr[illegible]ar of English and a considerable body of researchers are developing this today.

As soon as one seriously considers the project of working toward a correct and comprehensive theory of proof in English, he is quickly faced with a crucial consideration. Since a discourse grammar takes as a starting point a sentential grammar, and since a sentential grammar for English does not exist in anything like a complete form, it becomes clear that the project cannot be begun in a systematic fashion. This objection is well-taken but fortunately a reasonable substitute for a sentential grammar is available at least for the part of English used in mathematical proofs. As a result of centuries of logical analysis of mathematical discourse we now have formally defined symbolic languages which are sufficiently rich so that all of mathematical discourse can be symbolically states. Thus, we may choose

formal language into which to translate proofs and use the grammar of this formal language as the sentential grammar needed for the theory of proof. Taking this path our resultant theory of proof will necessarily be an idealization of an actual theory of proof in the same sense that, say, a formal language for arithmetic is an idealization of the part of English used in discourse about arithmetic. If it so happened that a group of mathematicians actually used a formal language in their investigations and they wrote their proofs in the formal language then we could investigate the body of proofs as such without translating and without regarding ourselves as developing an idealization.

Moreover, the use of the symbolic language may in the end be seen as a distinct advantage as it may enable the theory to transcend English and provide a theory of proof for other languages as well. However, one should not overlook the possibility that the idiosyncrasies of the various languages will also make themselves known on the discourse level and, in particular, in the proofs expressible in the various languages. This is not to suggest that a conclusion may be provable from certain premises in one language but not in another, though this may be true. Our suggestion was that even if exactly the "same" conclusions are provable from the "same" premises in two different languages it may turn out that there are means of doing it in one language not available to the other. Both of these hypotheses are likely -- and perhaps interesting to investigate.

--The Value of a Theory of Proof

Before we can consider the possible value of a theory of proof, we should try to determine specifications for a theory which could actually be developed. Otherwise, our speculations would be too hypothetical to be very interesting.

In the first place we postulate the existence of manageably small set of simple rules of inference which must be known in order, for example, to be able to prove the main theorems of plane geometry and arithmetic. It is immaterial whether these rules, which we will call the basic rules, are redundant. [A set of, say, three rules is redundant if everything that can be proved using all three can also be proved using only two.] We can easily imagine that the basic rules can be discovered. It is my opinion that the basic rules could be discovered and formulated within a short time by several logicians working with several high school mathematics teachers -- provided that the mathematics teachers (1) had been in the habit of making up new proofs and encouraging their students to make up new proofs and (2) had been developing geometry in different ways from year to year. In other words, the mathematics teachers working on the project must have some wide experience to refer to in these matters. What I have in mind is the situation wherein several linguists work with several native informants in developing a sentential grammar of an exotic language.

In order to discuss the value (utility) of a theory of proof then let us imagine that we have the basic rules

neatly formulated. Now, when we are asking about the value of this theory of proof what we are really concerned with is the possible answers to the following question: how could a mathematical educator use this theory to improve mathematical education?

A theory of proof which included the basic rules would provide strong (self-conscious) knowledge of the rules of inference commonly used in elementary mathematics. It seems to me that there are four areas within mathematical education in which such knowledge would be of use, viz. in teaching, in testing and guidance counseling, in curriculum design and in attempts to understand the psychology of mathematical learning.

Teaching: One important part of a mathematical education is learning to reason deductively and developing skill at it. There may be much more to learning to reason than merely acquiring knowledge and skill in the use of the rules--but certainly these are part of it. A teacher who knew the rules in the strong sense, i.e., he not only knew how to use them, but he also could refer to them explicitly, formulate them, etc.--such a teacher would be in a very advantageous position vis-a-vis trying to teach mathematical reasoning. Firstly, he would be better able to detect ignorance of specific rules. Now, when a teacher sees a student having difficulty with a proof he is left to his own devices as to what the difficulty is. Secondly, he would be able to be much more clear in his own writing of proofs

because he could be self-consciously critical of his own proofs. Thirdly, he would have a guide in choosing exercises and examples. When the class is having difficulty seeing a proof which involved a complicated application of a rule the teacher would be able to choose another theorem which involves a simpler application of the same rule, and then, in presenting it to the class he could point out that the reasoning in the complicated case is similar to the reasoning in the simple case. All three of these points hinge on the advantage that an articulate teacher has over one who is merely expert in the subject matter. Consider, for example, the excellent tennis player who is not articulate about what is involved in playing tennis. In trying to teach a beginner to play tennis, the expert player is reduced to showing. If he sees the student doing something wrong he cannot say exactly what is wrong. Even in showing the student what the motions are like, the teacher will not know what to exaggerate and he will not be able to distinguish his own idiocyncrasies from what is essential about tennis. Finally, he will be poor at developing drills, etc.

Testing and Guidance Counseling: It seems to me that a student's ability in deductive reasoning is an important index of his mathematical aptitude, his ability to learn mathematics. This means that a student who is skilled in understanding and producing mathematical proofs will be much more likely to benefit from mathematics courses than one who does not have such skills. It is obvious that a man

who has a characterization of what he wants to test is in a better position to design a test than a man who does not have such a characterization. A theory of proof is a characterization of the abstract structure underlying reasoning ability and it should provide a very useful framework for designing tests of reasoning ability. At the very least a theory of proof would provide a better knowledge of what is being measured in tests of reasoning ability ~~and~~, therefore, also in mathematical aptitude tests.

In order to get an idea of how such tests may be helpful in guidance counseling we must speculate concerning the kinds of things that might be discovered by use of the tests. For example, one might be able to show experimentally that unless a student had acquired (weak) knowledge of the basic rules by a certain age the chances of his ever being competent in mathematics are very slim. This would enable counselors to advise students concerning careers in mathematics and related areas. Moreover, it is not unreasonable to suppose that normal mathematical development could be characterized in terms of the number and kind of rules learned at various ages. This would permit objective identification of unusually able and unusually backward students again leading to more efficient and more scientific counseling. The professional mathematical educator can certainly conceive of other applications in this vein.

Curriculum Design: One of the aims of curriculum design is to trace a sequence of topics in mathematics

which parallels the optimal development of the students interests and abilities. The reason for this is the desire to give the student the maximum benefit from his formal educational experience. The idea is that the student is best educated by presenting to him at each stage in his education those concepts and proofs which he is best able to respond to. It is absurd either to present things which are too trivial or to present things that are beyond the student's ability. It seems to me then that a characterization of the development of mathematical skill in terms of the number and kind of rules acquired at various ages would provide a valuable framework for use in the design of an efficient curriculum. It would at least permit the knowledge of what would be very difficult and what would be very easy as far as reasoning is concerned and this, in turn, would permit more rational choices among alternative theorems to be presented or between alternative developments of a particular topic.

In addition one can easily imagine a battery of specific remedial programs each designed to teach a specific rule or sequence of rules. Such remedial programs used in conjunction with the diagnostic tests mentioned above might very well form a formidable weapon in trying to overcome inadequate preparation.

In the discussion of knowledge of rules of inference we suggested that complex rules are learned through experience with simpler ones. If this turns out to be true then the

details of the interrelation of knowledge of complex and simple rules will be very important in the choice of alternative developments of a subject as well as in the design of drills and so on.

Finally, we return to the hypothesis of active and passive vocabularies of rules. The truth of this hypothesis would lend additional justification to the suggestions of Professor J.J. LeTourneau (personal communication) to the effect that there should be two separate but parallel mathematics programs--one aimed at developing skill and concrete experience in creating theorems and proofs, the other aimed at acquainting the student with the body of existent mathematical knowledge. Naturally, a theory of the active vocabulary would be applied in the former, whereas the latter would use the passive theory.

Psychology: It is already clear enough that a theory of proof would provide a fruitful source of ideas for hypotheses and experiments in the psychology of mathematical learning. Moreover, one might wish to consider a more comprehensive theory of proof as an idealized description of the more-or- less behavioral aspects of the psychological processes of reasoning. We have already pointed out that the written (or spoken) proof is our only access to another person's reasoning processes. The written proof is a permanent record of the reasoning and, moreover, it is a "trace" of the behavioral aspect of the reasoning. The rules of inference in accordance with which the proofs are written are thus more-or-less behavioral "norms". Given all this, it is easy to speculate that a theory of proof could lead to

a psychological theory of deductive reasoning -- perhaps analogous to the way that Keplers Laws <u>describing</u> the orbits of planets lead to a kinetic theory <u>explaining</u> the orbits in terms of the effects of forces.

Finally, on the subject of applications of a theory of proof, I would like to suggest that the quality of writing of mathematics texts could be greatly improved if the writers would take the trouble to learn the rules of inference used by their prospective audiences. A mature mathematician must learn how to reason in a fashion understandable to a freshman if he wants freshmen to learn the mathematics (and not just memorize). Frequently, the mature mathematician encounters (in teaching) theorems which he sees "immediately" and he finds himself at a loss as to what to say to prove them. If he knew the rules of inference used by his class then he would know exactly what to say. If mathematics texts (and mathematics teaching) are improved in this way then one can expect that capable but non-genius students will be more able both to appreciate the beauty of mathematics and also to keep from "getting turned-off by the chicken scratching." Quite possibly all this could lead to the kind of improvement in the field of mathematics that we have seen after the re-discovery of the axiomatic method. In the axiomatic method we find the ideal of the deductive/definitional organization of branches of mathematics: a theory of proof provides a partial answer to the question of what deduction <u>is</u>.

Following all of these hopeful speculations I want to emphasize two negative points. In the first place, none of the above applications will be easily or mechanically achieved despite the fact that much of the groundwork is done. A tremendous amount of very detailed creative thought, dialogue and experimentation is needed. There is even cause to wonder whether there is a natural place to begin. And, there are terrifying pitfalls, one of which is the gap between the precision and simplicity of the symbolic languages, on the one hand, and the vagueness, ambiguity and complexity of natural language on the other. Anyone seriously desiring to pursue any of the above applications must become extremely sensitive to the nuances of normal English--and very few mathematicians have the patience for this. A pilot experiment in deductive reasoning recently conducted in a Philadelphia school ended distressingly because the subjects were diverted by too many linguistic red herrings in the test questions. Something can be perfectly clear in the symbolic language and perfectly confusing when translated mechanically into English.

Paradoxically, the second negative point issues from the exhilarating feeling of power and self-confidence that a mathematically competent person derives from learning to be articulate about what he is good at, i.e., from learning a clearly presented and apparently comprehensive theory of proof. Such a person naturally wants to teach the theory to his students--but if the students are not yet good

at reasoning they cannot appreciate the significance of what they are learning. They may learn the rules and they may learn how to follow the rules. The disaster is that they come to believe that mathematical reasoning <u>is</u> nothing but following rules. As we pointed out in the beginning of this part of the paper, if a person has his mind occupied with the rules then the chances are slim that he will have any attention left for the subject matter or for the deeper parts of reasoning. If a person learns the rules as external rules (as prescriptions) and not as descriptions of what he already does, the result is stultifying. If pressure is put on a student to accept a rule self-consciously before he knows the rule non-self-consciously, i.e., if a rule is imposed on a student, he will either rebel or lose his intellectual integrity or adopt the view that it's all a silly game. Another equally undesirable but less disastrous effect of teaching an uncomprehensive theory of proof even to students who can appreciate it derives from the fact that they may reason according to rules not in the theory. In this case, the students will tend not to use the rules absent from this theory thus weakening their powers of reasoning. The upshot is that they will be poorer at reasoning after learning the theory than they were before learning it.

PART III

Two Theories of Proof

This part of the paper has a dual purpose. In the first place we will discuss two kinds of theories of proof. The first kind will be called a theory of linear proof. The second has been called a theory of suppositional proof. The term "natural deduction" has often and correctly been used to refer to the second kind of theory but I shall not do so here because many of the theories so-called are not of the second kind -- they must be thought of either as disguised linear theories or theories of a third kind (see postscript below). The second purpose of this part is to develop some of the main ideas needed in constructing a comprehensive theory of proof. The reason for choosing the linear and suppositional theories for this purpose is because the linear theory includes only rules of a very simple nature and the suppositional theory can be seen as the result of making the linear theory more comprehensive.

--Theories of Linear Proof

Theories (or systems) of linear proof can be traced historically to Frege, who worked in the last century, and

perhaps even to Aristotle. A linear proof from a set of axioms is one in which each subsequent step either is an axiom or is derived immediately from axioms and/or previously proved lines. In other words a linear proof of c from P is written linearly in a column, say, beginning with the premises P at the top and proceeding step-by-step through intermediate conclusions all derived from P to c, the sentence to be proved. This is the idea, in practice things are a little more complicated, but the following general statement always holds -- in a linear proof from premises P to conclusion c each sentence in the proof is a logical consequence of P. (The reader should note that the concept of logical consequence as defined above is not relative to any system of proof.)

There are three minor modifications to be made to the above loose account of linear proofs. The first is that for clarity the assumptions shall be marked as assumptions to make it clear that they are not asserted to follow from any sentences which they may happen to follow. The second is that assumptions may be written at any place in the proof, not just at the top. The point here is that one may try to prove a theorem from only some of his axioms but then discover that he needs an additional one. The modification permits this to be written at the point needed rather than at the top. Finally, in addition to assumptions and inferences, properly so-called, all linear systems of proof permit the writing of so-called logical axioms at any point

in a proof. One prominent logical-axioms rule permits any proof to be lengthened by the addition of any identity (t=t).

Let us use the corner bracket (⌜) to mark assumptions. Thus, ⌜p could be read 'assume p'. Next we will give an example of a linear theory of proof all of whose rules are commonly used, perhaps most prominently in algebra. Following the statements of the rules will be a proof of $(x)(x=x^{-1-1})$ from the axioms of groups.

Rule Set A

Initial String Rules (Kernel Rules)

(1) Initial Assumptions: ⌜p is a proof

(2) Initial Logical Axioms: (t=t) is a proof

Production Rules

(3) Assumptions: any proof may be lengthened by the addition of ⌜p .

(4) Logical Axioms: any proof may be lengthened by the addition of any identity (t=t).

(5) Substitution of Equals: any proof containing (t=s) and also p may be lengthened by adding p' where p' is the result of replacing occurrences of t in p by s and/or vice versa.

(6) Instances: any proof containing (v) p(v) may be lengthened by adding p(t) - where v is a variable and t is a term composed of constants.

(7) Generalizations: any proof containing p(d), d a constant, may be lengthened by addition of (v)p(v) provided that no assumptions concern d (i.e., provided d is "arbitrary").

(8) Repetition: any proof may be lengthened by repeating any previous line.

Obviously each of the above rules corresponds exactly to a rule commonly used in proofs in algebra. Notice however that there are commonly used rules which do not appear in the list. For example the only way of instantiating is by rule 6 and this permits the elimination of quantifiers <u>only one per application</u>. This will be an annoying deficiency. Similarly for generalizations. Another deficiency is that substitutions can be done using only one equation at a time. In the proof below we have starred the lines that would remain were the deficiencies eliminated.

$$\vdash (x)(y)(z)(x\cdot(y\cdot z) = (x\cdot y)\cdot z) \qquad *$$

$$\vdash (x)(x\cdot 1 = x) \qquad *$$

$$\vdash (x)(1\cdot x = x) \qquad *$$

$$\vdash (x)((x\ x^{-1}) = 1 \qquad *$$

$$\vdash (x)((x^{-1}\cdot x) = 1 \qquad *$$

$$(y)(z)(a\cdot(y\cdot z) = (a\cdot y)\cdot z)$$

$$(z)(a\cdot(a^{-1}\cdot z) = (a\cdot a^{-1})\cdot z)$$

$$(a\cdot(a^{-1}\cdot a^{-1^{-1}}) = (a\cdot a^{-1})\cdot a^{-1^{-1}}) \qquad *$$

$$(a^{-1}\cdot a^{-1^{-1}}) = 1 \qquad *$$

$$a\cdot 1 = (a\cdot a^{-1})\cdot a^{-1^{-1}}$$

$$a\cdot a^{-1} = 1 \qquad *$$

$$a\cdot 1 = 1\cdot a^{-1^{-1}} \qquad *$$

$$a\cdot 1 = a \qquad *$$

$$a = 1\cdot a^{-1^{-1}}$$

$$1\cdot a^{-1^{-1}} = a^{-1^{-1}} \qquad *$$

$$a = a^{-1^{-1}} \qquad *$$

$$(x)(x = x^{-1^{-1}}) \qquad *$$

Having a more powerful instantiating rule would permit going from the associative law directly to the first unquantified line -- skipping two lines. The other two unstarred lines would be skipped by doing two substitutions at a time.

Incidentally the above rule set (or discourse grammar) describes proofs -- but it does not make explicit what "a proof of c from P" is. Naturally, we define a proof to be a proof of c from P if c is the last line of the proof and all assumptions in the proof are in P. The above example is a proof of $(x)(x=x^{-1-1})$ from the group axioms.

As the rule set is being used here, the (metalinguistic) symbols p, p(t) p(v), and p(d) refer to formulas in the language of groups. Thus this set of rules presupposed a sentential grammar for the language of groups. However, if we interpreted the symbols as referring to formulas in the arithmetic language, then we could use Rule Set A for the theory of proof needed to complete the Partial Grammar of the Arithmetic Language given at the end of Part I. This would actually be a bit silly for two reasons: first, the Partial Grammar has no quantifiers so rules 6 and 7 would never apply; second, the Partial Grammar <u>does</u> have the logical connectives whereas none of the rules permit any inferences involving connectives. The point, therefore, is <u>not</u> that the Partial Grammar would be finished but rather that the reader can now see what a finished grammar would be like. The respective natures of an alphabet, a rule set for words,

a rule set for phrases and a rule set for sentences are already clear from the Partial Grammar. Now we have also seen a discourse grammar which describes or produces a certain set of proofs. This discourse grammar, Rule Set A, is a theory of proof.

Rule Set A is obviously a correct theory of proof -- each of its rules corresponds exactly to (or is) an actual rule of inference that we have all used when doing proofs in elementary group theory. Rule Set A is obviously not comprehensive in the sense that I have defined the term because, e.g., it lacks the complex rules alluded to above which permit the unstarred lines to be omitted. However, it is complete in a certain sense.

A theory of proof for a particular language is called <u>equationally complete</u> when the following holds: given any set of equational sentences (either equations properly so-called or universal generalizations thereof) and any single equational sentence c, if c is a logical consequence of P, then there is a proof of c from P constructible by the rules of the theory. Rule Set A is equationally complete. This fact will be plausible to any reader who understands it. To the other readers the following remarks are addressed. Let P be the axioms for groups. Let c be any equational sentence written in the language of groups and which is true in all groups. c, then, is a logical consequence of P; since (1) a group is by definition any mathematical system in which the axioms of groups are true

and (2) to say that c is a logical consequence of P is to say that c is true in any mathematical system which makes all of the sentences in P true. The above-mentioned completeness condition implies, then, that by using Rule Set A one can construct a proof starting with P as assumptions (as in the example) and ending with c. In fact, such a proof can be gotten by lengthening the one given as a sample.

Incidentally, the equational completeness of Rule Set A was proved several years ago (by Dana Scott and Jan Kalicki).

In any theory of proof which describes or produces only linear proofs, it is possible to give a very simple description of all proofs from a particular set P of premises to a particular conclusion, c. Given a definition of the logical axioms and the rules one can then say: <u>a proof of P from c is a finite sequence of lines ending with c, each subsequent line of which either is an assumption in P or is a logical axiom or is obtained from previous lines by a rule</u>.

The underlined expression (or rather an even simpler version of it) has become a slogan and, sometimes, a battlecry. One eminent logician related to me that when he first heard this slogan presented he was struck by its simplicity and truth and was moved to say to himself, "By God, that <u>is</u> what proofs are!"

If one takes the slogan as a rough description of all proofs, then one is led (1) to distinguish three kinds of rules of inference and (2) to believe that all rules of inference must be of one of the three kinds. The first kind contains only the rule of <u>assumption</u> -- essentially to the effect that an assumption may be written to start or to lengthen any proof (provided that it is marked as an assumption). The second kind contains all <u>logical axiom rules</u> -- to the effect that a logical axiom may be written to lengthen any proof. The third kind contains all <u>immediate inference</u> rules; rules which state that any proof containing one or two (or some fixed finite number of) sentences of certain specified forms may be lengthened by adding a sentence in another form.

<u>--Immediate Rules and Subsidiary Proof Rules</u>

It so happens that by surveying the proofs in the mathematical literature (or by looking at our own proofs) we find many rules that are <u>not of</u> any of the above three kinds. Indeed, if all rules were of the three above kinds then there would be no room in mathematical reasoning for making subsidiary assumptions. Much of the most elegant and enlightening reasoning in mathematics turns on the ability to imagine good subsidiary assumptions. Below are some examples. (1) In proving that the square root of two <u>is not</u> rational we assume, in addition to the axioms of arithmetic, the subsidiary assumption that the square root

of two is rational. (2) In proving the right cancellation law $[(x)(y)(z)((x\cdot z = y\cdot z) \supset x=y)]$ from the group axioms we assume, in addition to the group axioms, that $a\cdot d=b\cdot d$ where a,b and d are arbitrarily chosen but fixed elements of the group. (3) Whenever we give proofs by cases after we have proved that there are two cases, say, we assume that the first case holds and then prove our theorem in that case, then we assume the second case and prove our theorem in that case -- finally we conclude that the theorem holds in general. . . . In each of these three examples the proof involves making subsidiary assumptions, assumptions other than those from which the conclusion is shown to follow.

At some point in each of these examples an inference is made not from certain previous lines in a proof but rather from (or on the basis of) a certain part of the proof. In other words, there are rules which can be stated as follows: any proof containing a subsidiary proof of a certain form may be extended by adding p. For example, in reductio reasoning we are following the rule: any proof containing a subsidiary proof beginning with p and containing a contradiction may be extended by adding $\sim p$ (not-p).

Consider the following proof of $\sim(x)\sim(x=x^{-1})$ from the group axioms which employs the above-stated reductio rule.

$\ulcorner (x)(y)(z)((x\cdot(y\cdot z)) = ((x\cdot y)\cdot z))$

$\ulcorner \quad (x)(x\cdot 1=x)$

$\ulcorner \quad (x)(1\cdot x=x)$

$\ulcorner \quad (x)(x\cdot x^{-1}=1)$

$\ulcorner \quad (x)(x^{-1}\cdot x=1)$

$$\left.\begin{array}{ll} \ulcorner (x)\sim(x=x^{-1}) & \\ \quad \sim(1=1^{-1}) & * \\ \quad 1\cdot 1^{-1}=1 & \\ \quad 1\cdot 1^{-1}=1^{-1} & \\ \llcorner \quad 1=1^{-1} & * \end{array}\right\} \text{subsidiary proof}$$

$\sim(x)\sim(x=x^{-1})$

The subsidiary proof is enclosed in matching brackets. The contradiction in question is "between" the starred lines. Notice that the conclusion is inferred <u>to follow from</u> the group axioms (not from all assumptions) on the basis of the subsidiary proof. Once a subsidiary proof is marked off by an ending bracket ($\llcorner$), it must be regarded as a separate unit in the proof. In particular, one may no longer apply any of the immediate inference rules to lines inside of the subsidiary proof. For example, we could not write down as a next line $\sim(1=1^{-1})$ by repetition because this does <u>not follow</u> from only the group axioms.

Let us use the phrase 'subsidiary proof rule' to refer to rules which permit the lengthening of a proof on the basis of a subsidiary proof. Of course, the most notorious of subsidiary proof rules is the rule of conditionalization which permits inference of 'if p then q' on

the basis of a subsidiary proof beginning with p and ending with q. We will give a proof of the right cancellation law from the group axioms to illustrate this. (In the proofs below we do not necessarily follow Rule Set A but use other commonly known rules as well.)

$$
\begin{array}{l}
\ulcorner (x)(y)(z)((x\cdot(y\cdot z)) = ((x\cdot y)\cdot z)) \\
\ulcorner \quad (x)(x\cdot 1=x) \\
\ulcorner \quad (x)(1\cdot x=x) \\
\ulcorner \quad (x)(x\cdot x^{-1}=1) \\
\ulcorner \quad (x)(x^{-1}\cdot x=1) \\
\qquad \left.\begin{array}{l}
\ulcorner \ a\cdot d=b\cdot d \\
\quad (a\cdot d)\cdot d^{-1}= (b\cdot d)\cdot d^{-1} \\
\quad a\cdot(d\cdot d^{-1})=b\cdot(d\cdot d^{-1}) \\
\llcorner \qquad a = b
\end{array}\right\} \text{Subsidiary Proof} \\
\qquad (a\cdot d=b\cdot d) \supset (a=b) \\
(x)(y)(z)(x\cdot z=y\cdot z \supset x=y)
\end{array}
$$

It will be valuable to notice that in proofs by cases more than one subsidiary proof is needed -- one for each case. Actually, all proofs-by-cases-rules are "combinations" of the two-case rule stated as follows: any proof containing 'c_1 or c_2', together with two subsidiary proofs one beginning with c_1 the other beginning with c_2 both ending with c, can be extended by adding c. To illustrate this we will give a proof of the two-sided cancellation law. The proof will involve one application of the two-case rule inside of a subsidiary proof on which conditionalization is used.

$(x)(y)(z)((x\cdot(y\cdot z)) = ((x\cdot y)\cdot z))$

$(x)\quad(x\cdot 1=x)$

$(x)\quad(1\cdot x=x)$

$(x)\quad(x\cdot x^{-1}=1)$

$(x)\quad(x^{-1}\cdot x=1)$

$(a\cdot d=b\cdot d)\ \mathrm{v}\ (d\cdot a=d\cdot b))$ -c_1 or c_2

$$\left.\begin{array}{l} a\cdot d=b\cdot d \\ (a\cdot d)\cdot d^{-1}= (b\cdot d)d^{-1} \\ a\cdot(d\cdot d^{-1})=b\cdot(d\cdot d^{-1}) \\ a=b \end{array}\right\}\ \text{first subdidiary proof}$$

$$\left.\begin{array}{l} d\cdot a=d\cdot b \\ d^{-1}\cdot(d\cdot a)=d^{-1}\cdot(d\cdot b) \\ (d^{-1}\cdot d)\cdot a=(d^{-1}\cdot d)\cdot b \\ a=b \end{array}\right\}\ \text{secondary subsidiary proof}$$

$a=b$ -cases rule*

$((a\cdot d=b\cdot d)\mathrm{v}(d\cdot a=d\cdot b))\quad a=b$ -conditionalization**.

$(x)(y)(z)(((x\cdot z=y\cdot z)\mathrm{v}(z\cdot x=z\cdot y))\quad x=y)$

The notations on the right are designed to help the reader see exactly where and how the two subsidiary proof rules are applied (* and **).

Before we proceed to a disucssion of theories of suppositional proof (theories involving subsidiary proof rules), the reader should note that the above three proofs are not linear because the subsidiary assumptions are not among the premises from which the proof proceeds. That is, for example, in the proof of the cancellation law from the group axioms

there are sentences which are not logical consequences of the group axioms. Thus in these proofs we do not reason in a linear fashion--we take "side trips".

--Theories of Suppositional Proof

The defining characteristic of a theory of suppositional proof is that the rules permit the use of subsidiary assumptions which are later "discharged" and are not among the assumptions from which the final conclusion is shown to follow. These rules are subsidiary proof rules which countenance an inference not from previous lines but rather on the basis of a subsidiary proof. Such rules are not unusual but rather they comprise the essence of clear, elegant mathematical reasoning. Indeed, I think the mathematically experienced reader will agree that linear proofs have a very computational flavor to them whereas suppositional proofs seem to embody more creative and enlightening reasoning.

There are a few questions concerning the formulation of suppositional rules which might have been annoying some readers. I will digress slightly at this point to take up some of them.

In the first place, there must be a rule for closing-out subsidiary proofs. This rule is stated as follows: If Π is a proof and Π contains more occurrences of $\ulcorner$ than of $\llcorner$ (more beginning brackets than ending brackets) then an ending bracket ($\llcorner$) can be written on the last line. The idea is that each assumption $\ulcorner p$ potentially starts a subsidiary

proof and that each subsidiary proof must start with the last assumption $\ulcorner p$ which is not already part of another subsidiary proof. Thus, each time an ending bracket is put in the proof there is exactly one beginning bracket with which to "match" it.

We define an occurrence of a sentence to be <u>closed</u> in a proof if it occurs between matched brackets, i.e., if it is bracketed. Otherwise it is called <u>open</u>.

A given proof is <u>a proof of its last line</u> (if open) <u>from its set of open assumptions</u>. Now we can state two important general principles for suppositional proofs.

Let p be a given line of a suppositional proof. (1) The sequence of lines up to and including p is itself a proof. Let us call this the subproof ending with p. (2) In any suppositional proof, each given line p is a logical consequence of the <u>open</u> assumptions of the subproof ending with p. (If p is prefixed by a $\ulcorner$, the $\ulcorner$ counts as in the subproof -- if by $\llcorner$ the $\llcorner$ does not count as in the subproof). Finally, we must point out that within a suppositional theory <u>the immediate rules must be stated so that they apply only to open lines</u>.

It is obvious that the framework of a suppositional theory is much more adequate for characterizing mathematical proofs than is the framework of a linear theory -- even though anything that can be proved in a given suppositional theory will also admit of proof in some linear theory.

In other words, we are not contrasting the abstract power of such theories but rather their relative adequacies in characterizing the proofs which we actually write. Given the advantage of suppositional theories we can ask: are there other kinds of rules of proof which could be added and which would constitute an even more adequate framework? Let us put this question another way. We have seen four kinds of rules of inference: (1) assumption rules, (2) logical axiom rules, (3) immediate inference rules and (4) subsidiary proof rules. Are there other kinds of rules which are actually employed in writing of proofs?

The most obvious kind of rule to suggest adding is a rule that permits the writing of "goals". Frequently when we are writing a proof, after some assumptions have been entered, we indicate our goal by writing, for example, "we want to show p". This is actually a very handy device which helps convey the reasoning to be expressed in the proof. Since the purpose of proofs is to express reasoning we should certainly consider such a rule. We could state it: Any proof may be lengthened by adding ?p. The question mark in this context could be read "to prove", say. We would then have to define all occurrences of ?p as closed because otherwise we would be applying immediate rules to what we were trying to prove -- thus begging the question.

Now let us consider another important kind of rule. We have actually given an example of this kind of rule, but

we did not classify it. Notice that all of the above kinds of rules apply only to a part of a proof to which they apply, i.e., it is not always necessary to look at each line in the whole proof in order to apply any of the above four kinds of rules -- assumption does not require looking at any lines, the same for logical axiom rules, immediate inference rules involve only fixed finite numbers of lines, subsidiary proof rules involve perhaps a few subsidiary proofs plus perhaps a few open lines. The rule of generalization, however, requires looking at a particular line p(d) and then checking through the whole proof to determine that nothing has been assumed about d -- i.e., that d is indeed arbitrary. Such rules we call <u>global immediate rules</u>. Thus, the classification of linear rules above was inadequate.

In addition there are subsidiary rules which involve reference to the entire proof to which they are applied. The most prominent example of a <u>global subsidiary rule</u> is the rule that is generally used in reasoning from an existentially quantified statement. For example, suppose that we have assumed the right cancellation law in a proof and we are aiming to prove $(\exists x)(y)(y\cdot x=x)\supset(x)(x=x^{-1})$. We assume the antecedent $(\exists x)(y)(y\cdot x=x)$ and then we say "let X_0 be such an object." ("Let" is a sure sign of an assumption.) We are assuming that x_0 is an arbitrary object satisfying the condition $(y)(y\cdot x_0=x_0)$. We reason then of an arbitrary b that $b\cdot x_0=x_0$ and that $b^{-1}\cdot x_0=x_0$. Then, using the cancellation law, infer $b=b^{-1}$. Since b is arbitrary, $(x)(x=x^{-1})$. Now we

say: "Since x_0 was arbitrary and $(x)(x=x^{-1})$ does not depend on x_0 the conclusion follows from the original assumption." This corresponds, in the below formalized version, to taking $(x)(x=x^{-1})$ out of the subsidiary proof and making it open [starred line].

$\ulcorner (x)(y)(z)(x\cdot z=y\cdot z \supset x=y)$

$?(\exists x)(y)(y\cdot x=x) \supset (x)(x=x^{-1})$

$\ulcorner (\exists x)(y)(y\cdot x=x)$

$\ulcorner (y)(y\cdot x_0=x_0)$ "let x_0 be such an object"

$b\cdot x_0=x_0$

$b^{-1}\cdot x_0=x_0$

$b\cdot x_0=b^{-1}\cdot x_0$

$b\cdot x_0=b^{-1}\cdot x_0 \supset b=b^{-1}$ (cancellation law)

$b=b^{-1}$

$\llcorner (x)(x=x^{-1})$

$\llcorner$ $(x)(x=x^{-1})$ *

$(\exists x)(y)(y\cdot x=x) \supset (x)(x=x^{-1})$

It might be worthwhile to do another example using the above rule. We will prove $(y)((\exists x)(Dx\&Hyx) \supset (\exists z)(Az\&Hyz))$ from $(x)(Dx \supset Ax)$.

$\ulcorner (x)(Dx \supset Ax)$

$\ulcorner (\exists x)(Dx\&Hbx)$

$\ulcorner$ $(Da\&Hba)$ "let a be such an object"

Da

$Da \supset Aa$

Aa

Hba

$Aa\ \&\ Hba$

$\llcorner (\exists z)(Az\&Hbz)$

$\llcorner (\exists z)(Az\&Hbz)$ *

$(\exists x)(Dx\&Hbx) \supset (\exists z)(Az\&Hbz)$

$(y)((\exists x)(Dx\&Hyx) \supset (\exists z)(Az\&Hyz))$

Because of limitations of space we merely mention a class of rules called definitional rules which actually form a subclass of the global subsidiary rules and which, as you can surmise from the name, countenance the use of definitions within proofs.

As a final question we consider the nature of an axiomatic development of a mathematical theory. An axiomatic development of a theory begins with the axioms. Subsequently the first theorem is proved, then the second, then the third , etc. However, after the first proof the axioms are not repeated. Moreover, in additinn to the axioms, previously proved theorems are also used as new axioms -- but these are generally not written down either. One way of characterizing such a development is to say that it is one long proof and that axioms and previously proved theorems can be used because they already appear above. There is something artificial about this characterization -- we usually say that a development of a theory contains many proofs, here we say that it is just one long proof. It has been suggested that further study might reveal a level above the level of proofs --a level containing "developments" composed of proofs. This suggestion implies that in a development of a theory there is structure which is not reducible to the structure of proofs.

--Summary of Suppositional Theories

We have seen that linear theories contain four kinds of rules: Assumption, logical axiom, immediate inference,

and global immediate inference. Next, we noticed that suppositional theories contain two additional kinds of rules: subsidiary proof rules and global subsidiary proof rules. In addition, we pointed out that the definitional rules are merely a species of the global subsidiary proof rules.

We explained the concept of an open sentence in a proof and we asserted that the general principle behind suppositional proofs has two parts: (1) that given a proof and a sentence p in the proof the part of the proof ending with p is also a proof (called the subproof ending with p) and (2) each such p is a logical consequence of the assumptions occurring open in the subproof ending with p. Given this principle, the notation for subsidiary proofs, and the classification of the rules, anyone having a background in mathematics is prepared to formulate his own theory of proof.

--Summary of the Paper

In part I we discussed some fundamental concepts involved in the analysis of mathematical reasoning. In addition, we introduced the concept of levels of language and pointed out that a grammar of an entire language should be composed of several grammars, one at each level. We also made the point that a proof is a certain kind of discourse which, in turn, suggested the possibility of a theory of proof--a discourse grammar which describes the proofs of a language.

In part II we outlined what a theory of proof would be like. We noted that the grammatical rules used in describing proofs are the rules of inference according to which we write proofs. We discussed the nature of our knowledge of rules of inference distinguishing weak and strong varieties of such knowledge. Finally, we speculated concerning the utility of a modest theory of proof vis-a-vis improvements in mathematical education.

In the course of Part III, we contrasted what has become the traditional theory with a newer and more adequate theory whose essential features were discovered in the 1920's (Jaskowski). The older theory holds that mathematical reasoning proceeds from axioms step-by-step to conclusions in a strictly linear fashion; i.e., each step in a proof must be a logical consequence of the axioms. This view was first systematized by Frege in the nineteenth century. It became the commonly accepted view until the 1920's when Lukasiewicz pointed out in his seminar that the theory did not agree with mathematical practice. Jaskowski, who was a student in the seminar, accepted the project of developing the exact details of a theory of proof which would take into account the salient features of mathematical reasoning not accounted for by Frege's theory. The newer theory is largely the result of Jaskowski's effort. The older theory we called linear, the newer suppositional.

We gave several examples of rules and proofs with the intention of supplying enough detail so that the basic ideas

can be grasped in a useful way.

--Postscript

The linguist and the logician will doubtless disagree with many of the above assertions. Several serious over-simplifications have been made--mostly concerning linguistics. My hope has been to show the overlap and possible cross-fertilization between, on the linguistic side, the ideas of Harris and Chomsky and, on the logical side, the ideas of Jaskowski. I have tried to do this in a way that would be of benefit to persons of diverse backgrounds. I was trying to write to an audience of mathematics educators, linguists, mathematicians, psychologists and logicians.

One final technical point: the so-called natural deduction systems found in books by Suppes, Lemmon, and Mates are not theories of suppositional proof. By looking carefully at each of them, one notices that the lines of their proofs are not sentences, but rather ordered pairs (P,c) where P is a set of "premises" and c is a single sentence. Moreover, a grammar to generate their proofs takes the form of a linear theory without any assumptions. In particular, in each of these systems each proof is a finite sequence of lines (P_1,c_1), (P_2,c_2),...,(P_n,c_n) where each subsequent line is either (axiomatically) of the form ({c},c) or else is the result of applying an immediate rule to a fixed, finite number of preceding lines. An example of such a rule would be: if (P_i,d) and $(P_j, d\supset c)$ are lines in a proof, then the proof can be lengthened by writing $(P_i \cup P_j, c)$. The idea

behind constructing a proof of c from P in these systems is not to try to deduce c from P, but rather to construct the ordered pair (P,c) starting initially from ordered pairs ({x},x) using rules which when applied to "valid arguments" produce "valid arguments." In a word, these systems stack-up valid arguments starting with the simple and building to the complex. As far as either the characterization of normal reasoning or utility in teaching is concerned, it seems to me that none of these systems fares well in comparison to a suppositional system as found in Johnstone and Anderson (1963), in Kalish and Montague (1964) or in Leblanc (1966).

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